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HARMONIC FORMS.

(SECOND PAPER.)

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Read in abstract before the Academy, at Emporia, Kan., November 30, 1907.

CHAPTER I.—PERFECT SQUARES.

THE requirements of a *magic square* are that all the columns, horizontal lines and two main diagonals of the proposed square should add equally.

In a *harmonic square* not only do the rectilineal lines and diagonals add equally, but the sum of the vertices of all possible regular quadrilateral figures, as squares, rectangles, rhombs, and rhomboids, add equally. Harmonic squares are possible only when the root of the square, or number of cells on a side, is divisible by 4.

The term *perfect square* is applied to a square which adds equally not only in all the rectilineal lines but also in all the diagonal lines, thus making in straight lines a number of sums equal to four times the number of cells on one side of the given square. Thus a square of 4 should give 16 equal sums; a square of 5 should give 20 equal sums; a square of 6 should give 24 equal sums, etc. But the perfect squares here shown are of a superior character, and not only add equally in the many ways shown, but also add equally in all possible quadrilateral figures in any part of the square when as many cells are included in the quadrilateral as the number of cells on a side of the square.

Perfect squares are now constructed of any number of cells on a side above 3. A perfect square of 3 is impossible, for the reason that the number of sums necessary to entitle a square of 3 to be called "perfect" is twelve, while the greatest number of equal sums that can be obtained from any three numbers of a regular series of nine numbers is eight, as shown in part I of this paper, published in volume XIV of these Transactions (1894), pages 47, 48.

To form any sort of a magic square the given series must be divided into as many sets as there are cells on one side of the square. Whatever arrangement be given to the first set must be followed absolutely by each of the other sets. The members of the first set need not be taken consecutively from

the series, but may be taken in any order whatsoever. An unused number of one set may be taken as a member of another set.

In odd squares all sets must be placed parallel to the first. In even squares compensatory arrangements must be observed so as to preserve a rhythmic or balanced effect. The initial members of the several sets in a perfect square need not be equidistant mathematically. It is only necessary in odd squares that the initials of the several sets bear the same relation to each other in position as the second does to the first, and in even squares be opposed, so as to balance.

SECTION 1.—PERFECT SQUARE OF FOUR.

Perfect squares of four cells on each side may be formed, as are harmonic squares, according to certain schemes which are here shown. They may be formed without the aid of a visible scheme; but human ability to see a mental picture of all numbers in position before writing any is not great and the visible scheme is a great help in that direction.

Before showing any of the schemes a few definitions would seem to be desirable.

DEFINITIONS.

An *adjacent* number, line or column is the one next to it in the same half square, as first and second are adjacent to each other; third and fourth are adjacent. Second and third, though contiguous, are not adjacent. An adjacent quarter is one on the same side, whether vertically or horizontally.

An *alternate* number, cell, line or column is the second removed, or with one intervening, as first and third are alternate; second and fourth are alternate.

An *opposite* line or column is the one in the opposite part of the square that would come against it if we fold the square along its middle line, as first and fourth are opposite; second and third, though contiguous, are opposite. An opposite cell or quarter is the one diagonally or diametrically opposite.

A *couplet* is two numbers in succession in a series, consisting, in a series whose first term is 1 and whose common difference is 1, of an odd and an even number; and in any other series consists of two numbers side by side when the entire series is arranged in pairs from the beginning. The first member of each couplet may be called the antecedent or leader and the second the consequent or follower.

A *pair* is two numbers standing side by side in the same

quarter of the square. A columnar pair are those in the same vertical line; a linear pair are in the same horizontal line. A couplet should never be a pair.

A *complement* of a number in any square is the difference between that number and half the sum of a line in that square.

SCHEDULE OF SCHEMES.

SCHEME I (fig. 17).—*Coupling-arrows joining alternate columns and opposite lines.* Numbers of trial arrangements govern two upper lines. This places 15 adjacent to 1 in same column.

In order to produce a perfect square from scheme 1, the eight coupling-arrows of the scheme are numbered at their upper ends in such a way that each of the two rows of arrow numbers adds 18 horizontally and 9 vertically. There are six possible combinations, each of which will produce a perfect square differing from every other. From these trial arrangements the following six model perfect squares are produced, strictly according to the scheme:

FIRST ARRANGEMENT.

1	7	6	4
8	2	3	5

1	14	11	8
15	4	5	10
6	9	16	3
12	7	2	13

No. 1.

SECOND ARRANGEMENT.

1	7	4	6
8	2	5	3

1	14	7	12
15	4	9	6
10	5	16	3
8	11	2	13

No. 2.

THIRD ARRANGEMENT.

1	6	7	4
8	3	2	5

1	12	13	8
15	6	3	10
4	9	16	5
14	7	2	11

No. 3.

FOURTH ARRANGEMENT.

1	6	4	7
8	3	5	2

1	12	7	14
15	6	9	4
10	3	16	5
8	13	2	11

No. 4.

FIFTH ARRANGEMENT.

1	4	7	6
8	5	2	3

1	8	13	12
15	10	3	6
4	5	16	9
14	11	2	7

No. 5.

SIXTH ARRANGEMENT.

1	4	6	7
8	5	3	2

1	8	11	14
15	10	5	4
6	3	16	9
12	13	2	7

No. 6.

Equal sums are obtained from this square by adding together four numbers in each of the following regular ways:

1.—By lines—	<i>Ways.</i>
Each of the columns (fig. 1).....	4
Each of the lines (fig. 2).....	4
Every diagonal line, entire or broken (figs. 3, 4).....	8
2.—By squares—	
The four corners and the four centrals (fig. 5).....	2
Any four contiguous numbers in a square (figs. 5, 6, 7).....	8
Corresponding corners of quarters (fig. 8).....	4
3.—By rectangles—	
Linear pairs opposite lines (figs. 9, 10, 12).....	3
Columnar pairs in opposite columns (figs. 9, 11).....	3
4.—By rhomboids and trapezoids—	
Columnar pairs at opposite ends of alternate columns (fig. 13),	4
Linear pairs at opposite ends of alternate lines (fig. 14).....	4
Opposite numbers in alternate lines (fig. 15).....	4
Opposite numbers in alternate columns (fig. 16).....	4
Total regular ways of adding 34.....	52

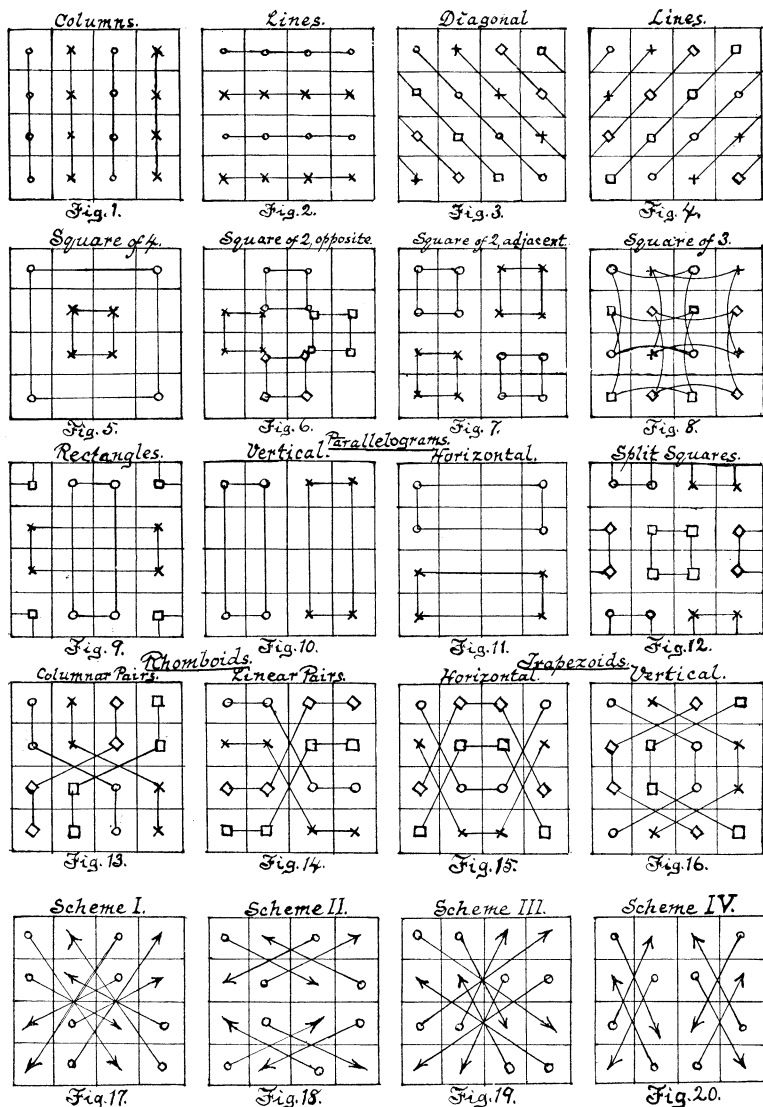
Each of these squares yields the equal sum of 34 in 52 regular ways, as here shown, taking the first one as an example:

RIGHT LINES.	RECTANGLES.	RHOMBOIDS.
1 + 14 + 11 + 5 = 34	1 + 14 + 15 + 4 = 4	1 + 15 + 16 + 2 = 34
15 + 4 + 5 + 10 = 34	14 + 11 + 4 + 5 = 34	14 + 4 + 3 + 13 = 34
6 + 9 + 16 + 3 = 34	11 + 8 + 5 + 10 = 34	11 + 5 + 6 + 12 = 34
12 + 7 + 2 + 13 = 34	1 + 15 + 8 + 10 = 34	8 + 10 + 9 + 7 = 34
1 + 15 + 6 + 12 = 34	15 + 4 + 6 + 9 = 34	1 + 14 + 16 + 3 = 34
14 + 4 + 9 + 7 = 34	4 + 5 + 9 + 16 = 34	15 + 4 + 2 + 13 = 34
11 + 5 + 16 + 2 = 34	5 + 10 + 16 + 3 = 34	6 + 9 + 11 + 8 = 34
8 + 10 + 3 + 13 = 34	15 + 6 + 10 + 3 = 34	12 + 7 + 5 + 10 = 34
DIAGONAL LINES.	TRAPEZOIDS.	
1 + 4 + 16 + 13 = 34	1 + 8 + 9 + 16 = 34	
14 + 5 + 3 + 12 = 34	15 + 10 + 7 + 2 = 34	
11 + 10 + 6 + 7 = 34	6 + 3 + 14 + 11 = 34	
8 + 15 + 9 + 2 = 34	12 + 13 + 4 + 5 = 34	
1 + 10 + 16 + 7 = 34	1 + 12 + 5 + 16 = 34	
14 + 15 + 3 + 2 = 34	14 + 7 + 10 + 3 = 34	
11 + 4 + 6 + 13 = 34	11 + 2 + 15 + 6 = 34	
8 + 5 + 9 + 12 = 34	8 + 13 + 4 + 9 = 34	
	Total sums of 34 = 52	

RULE.—To fill a square according to this scheme consider the eight numbers in one of the trial arrangements to represent the eight couplets of the series; apply the numbers in consecutive order to the upper ends of the coupling-arrows in the scheme; double each number in turn and place the product in the cell corresponding with the point of the arrow; subtract 1 from that product and place in the cell corresponding to the tail of the arrow (represented by a circle). The result will be a perfect square.

From each of these six squares fifteen other squares ap-

parently different may be formed by simply transposing one or more lines from any one side of the square to the opposite side, without in any degree changing the perfection of the square. This makes ninety-six perfect squares that can be made from that one scheme.



The construction of schemes for perfect squares is hedged by the necessity of having not more than two arrow-points in

any line, column, or diagonal, whether whole or broken. And besides, an arrow cannot begin and end in the same line, either vertically, horizontally, or diagonally. Only four schemes for perfect squares have so far been constructed.

SCHEME II (fig. 18).—*Coupling-arrows joining alternate columns and adjacent lines.* This scheme is readily transformable from scheme I. It is only necessary to transpose the second and fourth lines. The numbers of the coupling-arrows in the trial arrangements are placed as before; but, instead of governing adjacent lines as before, they are to be applied to the top and bottom lines. This puts 15 in the lower left-hand corner. The model squares resulting from the six arrangements are as follow:

FIRST ARRANGEMENT.

1	7	6	4
8	2	3	5

1	14	11	8
12	7	2	13
6	9	16	3
15	4	5	10

No. 7.

SECOND ARRANGEMENT.

1	7	4	6
8	2	5	3

1	14	7	12
8	11	2	13
10	5	16	3
15	4	9	6

No. 8.

THIRD ARRANGEMENT.

1	6	7	4
8	3	2	5

1	12	13	8
14	7	2	11
4	9	16	5
15	6	3	10

No. 9.

FOURTH ARRANGEMENT.

1	6	4	7
8	3	5	2

1	12	7	14
8	13	2	11
10	3	16	5
15	6	9	4

No. 10.

FIFTH ARRANGEMENT.

1	4	7	6
8	5	2	3

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

No. 11.

SIXTH ARRANGEMENT.

1	4	6	7
8	5	3	2

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

No. 12.

Ninety-six additional perfect squares, in every respect equal with the original ninety-six, can be made from these six arrangements.

SCHEME III (fig. 19).—*Coupling-arrows joining alternate*

lines and opposite columns. This scheme is the same as scheme I turned over on its direct central diagonal. The numbers of the coupling-arrows in the trial arrangements are to be placed in two columns so that they add 18 vertically and 9 horizontally, and are to be applied to the left-hand ends of the arrows, which fall in the first and second columns. That puts 15 at the top of the second column adjacent to 1. The resulting model squares here follow:

FIRST ARRANGEMENT.

1	8
7	2
6	3
4	5

1	15	6	12
14	4	9	7
11	5	16	2
8	10	3	13

No. 13.

SECOND ARRANGEMENT.

1	8
7	2
4	5
6	3

1	15	10	8
14	4	5	11
7	9	16	2
12	6	3	13

No. 14.

THIRD ARRANGEMENT.

1	8
6	3
7	2
4	5

1	15	4	14
12	6	9	7
13	3	16	2
8	10	5	11

No. 15.

FOURTH ARRANGEMENT.

1	8
6	3
4	5
7	2

1	15	10	8
12	6	3	13
7	9	16	2
14	4	5	11

No. 16.

FIFTH ARRANGEMENT.

1	8
4	5
7	2
6	3

1	15	4	14
8	10	5	11
13	3	16	2
12	6	9	7

No. 17.

SIXTH ARRANGEMENT.

1	8
4	5
6	3
7	2

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

No. 18.

Remark.—It is to be noted that the celebrated square of the Greek Moschopulus, first published, so far as known, in the sixteenth century, translated into Latin by Delahire and read by him before the French Academy of Sciences in 1691, is really a harmonic square and fits nicely into my harmonic sys-

tem as square No. 41, under scheme II, published on page 58 of volume XIV, Transactions Kansas Academy of Science (1893-'94), and by an easy transposition of the fourth column for the third, and the fourth line for the third, it becomes a perfect square like No. 15 above.

SCHEME IV (fig. 20).—*Coupling-arrows joining alternate lines and adjacent columns.* This is the same as scheme III, with the second and fourth columns transposed. The numbers in the trial arrangements are therefore to be applied to the first and fourth columns. This puts 15 in the upper right-hand corner. The six model squares here follow:

FIRST ARRANGEMENT.

1	8
7	2
6	3
4	5

1	12	6	15
14	7	9	4
11	2	16	5
8	13	3	10

No. 19.

SECOND ARRANGEMENT.

1	8
7	2
4	5
6	3

1	8	10	15
14	11	5	4
7	2	16	9
12	13	3	6

No. 20.

THIRD ARRANGEMENT.

1	8
6	3
7	2
4	5

1	14	4	15
12	7	9	6
13	2	16	3
8	11	5	10

No. 21.

FOURTH ARRANGEMENT.

1	8
6	3
4	5
7	2

1	8	10	15
12	13	3	6
7	2	16	9
14	11	5	4

No. 22.

FIFTH ARRANGEMENT.

1	8
4	5
7	2
6	3

1	14	4	15
8	11	5	10
13	2	16	3
12	7	9	6

No. 23.

SIXTH ARRANGEMENT.

1	8
4	5
6	3
7	2

1	12	6	15
8	13	3	10
11	2	16	5
14	7	9	4

No. 24.

Remark.—One of the most conspicuous objects in an old copper engraving, entitled "Melancholia" and executed by Al-

brecht Dürer in Nuremberg, Bavaria, in 1514, is a well-prepared magic square. It appears to be a modification of the square of Moschopolus, by *reinversion*, or reversing and inverting, and transposition of the second and third columns. The date, 1514, however, would seem to antedate or be cotemporary with Moschopolus. The square is perfectly harmonic; and on reinversion is exactly the same as my No. 47, scheme IV, harmonic system, page 59 of volume XIV, Transactions Kansas Academy of Science, 1894. On transposing the third and fourth columns and the third and fourth lines it becomes "perfect" and is No. 21 above. A comparison of Nos. 21 and 15 above will show the similarity of those two old magic squares when reduced to a primary condition. They are both probably modifications of some still older magic square.

As each of the above twenty-four primary squares can be transposed into fifteen other squares, it follows that 24×16 , or 384, different perfect squares can be constructed from these four schemes. No other schemes for perfect squares of 4 are possible.

A SIMPLE METHOD.

A simple method of forming perfect squares of 4 without the aid of a scheme or prearranged plan has been evolved by Mr. D. H. Davison, of Minonk, Ill.; and as still further simplified and modified by the present author is here presented:

1			15
	2	16	

FIRST STEP.

1			15
14			4
	2	16	
	13	3	

SECOND STEP.

1	12	6	15
14			4
11	2	16	5
	13	3	

THIRD STEP.

1	12	6	15
14	7	9	4
11	2	16	5
8	13	3	10

COMPLETE SQUARE.

No. 19.

1.—(a) Place 1 in the upper left-hand corner and (b) its complement, 16, in the alternate or second cell diagonally from it; (c) place 2 in either one of the four cells next to 16, above, below, or on either side, and (d) its complement, 15, in the alternate cell diagonally from it.

2.—(a) Place 3 in one of the three remaining cells next to 16, and (b) its mate, 4, in a cell relatively from 3 as 2 is from 1. For instance, if 2 is in an alternate line and adjacent column from 1, then 4 must be placed in an adjacent column and alternate line from 3. Thus the harmony of the square is preserved. (c and d) Their complements, 14 and 13, are to be

placed in the alternate diagonal cells from each of them respectively.

3.—(a) Place 5 in either one of the two remaining cells beside 16, and (b) its mate, 6, in the same position relatively to 5 as 2 is to 1; that is to say, in an alternate line and adjacent column. (c and d) Their complements, 12 and 11, are to be placed, as before, in the alternate diagonal cells from each of them.

4.—(a) Place 9 in the only remaining cell next to 16, and (b) its mate, 10, in the same relative position to it as 2 is to 1. (c and d) Their complements, 8 and 7, are to be placed as before in the alternate cells diagonally from them.

The numbers 2, 3, 5 and 9 are always to be laid around 16. The reason for this is that these four places are just a "knight-step" or paladin step from 1; all other cells in the square are in the same line with 1, either vertically, horizontally, or diagonally; 16 is at the intersection of the two diagonals. The four numbers mentioned must occupy those four cells and no other. They may be laid in any order. The number 9 may be placed first if preferred, and may be placed in any of the four vacant cells next to 16. It is immaterial what order these four numbers are placed in; it *is* material where their mates are placed.

Once an antecedent number is placed in a cell there is only one place for its mate, according to the scheme, and one place for each of their complements. There are really four places in either of which a consequent of a couplet may be placed; but there are four schemes; and whatever scheme is adopted for the first couplet must be followed for all the rest, in order to preserve the unity and harmony of the square.

So the placing of 2 immediately predetermines the position of seven other numbers; for the position of every consequent of a couplet must bear the same relation to its antecedent that 2 bears to 1. Mates are always a paladin step apart, that is, two cells in one direction and one cell at right angles to the two; for while they may appear otherwise, the one is a paladin step from the other across the margin of the square, as though the other were in its proper position in an adjoining square. The real position of the consequents, however, depends upon the positions occupied by 3 and 5, in the placing of each of which there is some latitude.

Similarly, the placing of 3 predetermines the position of

three other numbers which are the leaders of the second couplet of the three following sets, as 3 is the leader of the second couplet of the first set.

The placing of any number immediately determines the position of its complement; and this complement can and should be laid at once, thus simplifying the construction of the square.

The first number laid around 16 has a choice of four positions; the second number has a choice of three; the third number has a choice of two positions and may be laid in either; the fourth number has only one place left for it. Multiplying together these factors, 4, 3, 2, and 1, we obtain 24 as the possible ways of arranging the sixteen numbers into a perfect square, with 1 placed in a certain definite place, as, for example, in the upper left-hand corner.

VARYING SERIES AND SPECIAL SUMS.

It is by no means necessary in order to produce a perfect square that a series shall be absolutely uniform; it is only necessary that the common differences shall be harmonic, rhythmic, or concordant. There are four kinds of differences in a series of sixteen terms, namely:

1. The difference between the antecedent and consequent in each couplet. There are eight couplets in a series and therefore eight of these differences ($=d$).

2. The differences between the leading couplet and the following couplet of each set, thus coming in the middle of each of the four sets. There are four of these in any series of sixteen numbers. For the purpose of distinction these differences will be called *notches* ($=n$).

3. The differences between adjacent sets of four in each half of the series. There are two such differences, one coming in the middle of each half. These will be called *side gaps* ($=g$).

4. The difference between the two halves of a series. There is but one of this. It is the *main gap* ($=G$).

Some other terms may be represented by letters for convenience. For example, let a represent the first term, q the last term, and S the sum of all the numbers in one row of the square, whether columnar, linear or diagonal.

On arranging the terms of a series of sixteen numbers in a row by number, the differences will appear thus:

$a \ 1 \ d \ 2 \ n \ 3 \ d \ 4 \ g \ 5 \ d \ 6 \ n \ 7 \ d \ 8 \ G \ 9 \ d \ 10 \ n \ 11 \ d \ 12 \ g \ 13 \ d \ 14 \ n \ 15 \ d \ 16 \ a$

It will be evident that in a square of 4, since four numbers are included in each equal sum, the sum of any line should equal twice the sum of the extremes (a and q); and the last term should equal the first term plus all the differences between the extremes. Written out in the form of equations these statements would appear thus:

$$\begin{aligned} S &= 2(q+a) \quad \text{or} \quad S = 2q + 2a, \\ q - a &= 8d + 4n + 2g + G, \quad \text{and} \\ q &= a + 8d + 4n + 2g + G; \quad \text{whence,} \\ S &= 4a + 16d + 8n + 4g + 2G \end{aligned}$$

A few examples will show how to use these formulæ:

PROBLEM.—Let it be required to write a square of 4 from a varying series which will give sums of 64.

SOLUTION.—There are eighty-three series of integral numbers, without resorting to fractions, which will yield sums of 64 in a square of 4. Three such series will be selected as examples:

(1)	1	2	5	6	9	10	13	14	18	19	22	23	26	27	30	31
(2)	3	5	6	8	10	12	13	15	17	19	20	22	24	26	27	29
(3)	4	5	7	8	9	10	12	13	19	20	22	23	24	25	27	28

In the first selection it will be seen that $a = 1$, $d = 1$, $n = 3$, $g = 3$, and $G = 4$; whence $2a = 2$, $8d = 8$, $4n = 12$, $2g = 6$, and $G = 4$. Total = 32.

In the second selection $a = 3$, $d = 2$, $n = 1$, $g = 2$, and $G = 2$; whence $2a = 6$, $8d = 16$, $4n = 4$, $2g = 4$, and $G = 2$. Total = 32, as before.

In the third selection $a = 4$, $d = 1$, $n = 2$, $g = 1$, and $G = 6$; whence $2a = 8$, $8d = 8$, $4n = 8$, $2g = 2$, and $G = 6$. Total = 32, half of 64.

Putting these series into squares of 4 it will be seen that they are just as perfect with these varying series as any regular series would make.

(1) $a=1, d=2, n=3, g=3, G=4.$					(2) $a=3, d=2, n=1, g=2, G=2.$					(3) $a=4, d=1, n=2, g=1, G=6.$				
64					64					64				
1	27	22	14	64	3	27	12	22	64	4	23	10	27	64
23	18	2	26	64	26	8	17	13	64	25	12	19	8	64
10	18	31	5	64	20	10	29	5	64	22	5	28	9	64
30	6	9	19	64	15	9	6	24	64	13	24	7	20	64
64	64	64	64	64	64	64	64	64	64	64	64	64	64	64
No. 25.					No. 26.					No. 27.				

ODD SUMS.

Perfect squares yielding odd sums cannot be constructed of integral numbers. A half-unit must be added to at least half of the numbers. This can be done in various ways, as (1) by inserting a half-unit in the main gap; (2) by inserting a half-unit in each of the side gaps and main gap; (3) by inserting a half-unit in each of the four notches and main gap; (4) by inserting a half-unit in each of the four notches, two side gaps and main gap; (5) by inserting a half-unit in each of the eight differences and main gap; (6) by inserting a half-unit in each of the eight differences, two side gaps and main gap; (7) by inserting a half-unit in each of the eight differences, four notches and main gap; and (8) by inserting a half-unit in each of the eight differences, four notches, two side gaps and main gap. In either case half the terms will be integral and half of them fractional. This principle is clearly seen in the three following examples, where irregular series beginning with 7 are arranged to add 77 in every direction.

(1)	7	8	10	11	14	15	17	18	20.5	21.5	23.5	24.5	27.5	28.5	30.5	31.5
(2)	7	9	10	12	13.5	15.5	16.5	18.5	20	22	23	25	26.5	28.5	29.5	31.5
(3)	7	8.5	10	11.5	14	15.5	17	18.5	20	21.5	23	24.5	27	28.5	30	31.5
(4)	7	8.5	10	11.5	14.	15	17	18.5	20	21.5	23	24.5	27	28.5	30	31.5

In order to facilitate inspection of the series and the placing of the several terms in a square the series should be arranged each in four sets, thus:

(1)				(2)				(3)				(4)			
7	8	10	11	7	9	10	12	7	8.5	10	11.5	7	10	14	17
14	15	17	18	13.5	15.5	16.5	18.5	14	15.5	17	18.5	8.5	11.5	15.5	18.5
20.5	21.5	23.5	24.5	20	22	23	25	20	21.5	23	24.5	20	23	27	30
27.5	28.5	30.5	31.5	26.5	28.5	29.5	31.5	27	28.5	30	31.5	21.5	24.5	28.5	31.5

No. 4 is the same series as No. 3 but differently arranged.

In the first series $a = 7$, $d = 1$, $n = 2$, $g = 3$, $G = 2.5$; which makes $2a = 14$, $8d = 8$, $4n = 8$, $2g = 6$, $G = 2.5$; total, 38.5, which is one-half of 77.

In the second series $a = 7$, $d = 2$, $n = 1$, $g = 1.5$, $G = 1.5$; therefore $2a = 14$, $8d = 16$, $4n = 4$, $2g = 3$, $G = 1.5$; total, 38.5.

In the third series $a = 7$, $d = 1.5$, $n = 1.5$, $g = 2.5$, $G = 1.5$; from which $2a = 14$, $8d = 12$, $4n = 6$, $2g = 5$, $G = 1.5$; total, 38.5, as before.

In another arrangement (4) of the third series $a = 7$, $d = 3$, $n = 4$, $g = -8.5$, $G = 1.5$; which gives $2a = 14$, $8d = 24$, $4n = 16$, $2g = -17$, $G = 1.5$.

From these series are constructed the following perfect squares:

(1) $a=7, d=1, n=2, g=3,$ $G=2.5,$					(2) $a=7, d=2, n=1, g=1.5,$ $G=1.5,$					(3 and 4) $a=7, d=1.5, n=1.5, g=2.5,$ $G=1.5,$				
7	30.5	15	24.5	77	7	25	16.5	28.5	77	7	30	11.5	28.5	77
28.5	11	20.5	17	77	18.5	26.5	9	23	77	24.5	15.5	20	17	77
23.5	14	31.5	8	77	22	10	31.5	13.5	77	27	10	31.5	8.5	77
18	21.5	10	27.5	77	29.5	15.5	20	12	77	18.5	21.5	14	23	77
77	77	77	77	77	77	77	77	77	77	77	77	77	77	77
No. 33.					No. 34.					Nos. 35 and 36.				

Two arrangements are shown for the third series. One (3) is laid according to model square No. 15, *ante*, and the other (4) is laid according to model square No. 19. Both are precisely alike; the only difference is in the order that the terms are taken. Some of the others may also be taken in different orders.

As shown in the preceding pages every sum of a row in any square of 4 is equal to twice the continued sum of double the first term of the series plus eight times the difference between the first and second terms, plus four times the difference between the second and third terms, plus twice the difference between the fourth and fifth terms, plus the difference between the eighth and ninth terms. Using the same letters as before to represent these several differences, the formula becomes, as already shown,

$$S = 2(2a + 8d + 4n + 2g + G).$$

To obtain 111 as the sum in a perfect square, as for any other number, we take half the amount, in this case 55.5, divide it up to suit fancy or convenience by making the several

111					DIFFERENCES.		SERIES.			
1	48	23	39	111	$2a=2$	$a=1$	1	3	7.5	9.5
25	37	3	46	111	$8d=16$	$d=2$	16.5	18.5	23	25
					$4n=18$	$n=4.5$	30.5	32.5	37	39
32.5	16.5	54.5	7.5	111	$2g=14$	$g=7$	46	48	52.5	54.5
52.5	9.5	30.5	18.5	111	$G=5.5$	$G=5.5$				
					55.5					
111	111	111	111	111						
No. 37.										

quantities used as differences represent any amount advisable or available, add the values of the several quantities, then proceed to lay out the series in strict accordance with the formula. The square may then be built up according to any one of the twenty-four model squares already shown. In the first example shown above, if we make $2a = 2$, $8d = 16$, $4n = 18$, $2g = 14$, and $G = 5.5$, then will $a = 1$, $d = 2$, $n = 4.5$, $g = 7$, and $G = 5.5$; total, 55.5.

				111				
20	34.5	25	31.5	111				
				111				
33.5	23	28.5	26	111				
				111				
30.5	24	35.5	21	111				
				111				
27	29.5	22	32.5	111				
111	111	111	111	111				

DIFFERENCES.				SERIES.			
$a=20$	$2a=40$	20	21	22	23		
$d=1$	$8d=8$	24	25	26	27		
$n=1$	$4n=4$	28.5	29.5	30.5	31.5		
$g=1$	$2g=2$	32.5	33.5	34.5	35.5		
$G=1.5$	$G=1.5$						
	55.5						

No. 38.

In the second example, in order to differ materially from the other, take 20 for the first term and let each of the other differences be 1 except the difference in the main gap (G), which will be 1.5. From these we obtain $2a = 40$, $8d = 8$, $4n = 4$, $2g = 2$, and $G = 1.5$. A series of sixteen terms prepared from these differences give the numbers as above, from which a perfect square adding 111 in all its parts may readily be constructed.

THE CIRCLE SQUARED.

Shall we essay the problem that has engaged the most eminent mathematicians for thousands of years, that of squaring the circle? But we perform the operation in an entirely new way, by a method that has never before been tried. No claim is here made that the problem is solved. It is entirely a play upon words. But if taking the diameter and circumference of a circle and placing the divisions in the form of a square, so that by addition of the parts in any direction the same circumference is obtained, if that is not squaring the circle it certainly is not circling the square. In other words, if it is not a circular square or a square circle it must be a circle squared, so it amounts to the same thing.

Take the figures that represent the circumference when the diameter is 1.0000, or as near that amount as four decimal places will give us, namely, 3.1416; though any other number

of decimal places might as well be taken, except that the farther the decimal is extended the more time and space it will take.

First we take half of our number, namely, 1.5708, and divide it up as we please according to the formula, being careful to make our several numbers sum up exactly our required number. Then, if we have taken $2a$, $8d$, $4n$, etc., to make our numbers correspond with, we should select numbers that are divisible by 2, 8, etc., in order to avoid fractional numbers when not necessary, as in the case where the sum of a line is odd.

Here are two examples. In the first one the series begins with the decimal surplus above 3 included in the circumference of the circle; the sum of the eight minor differences ($8d =$

THE CIRCLE SQUARED.				3.1416				DIFFERENCES.		SERIES.			
1.0000	.7345	.9345	.4726	3.1416		$2a =$	.2832	$a =$	.1416	.1416	.2398	.3053	.4035
1.2655	.1416	1.3310	.4035	3.1416		$8d =$	.7854	$d =$	.0982	.4726	.5708	.6363	.7345
						$4n =$	.2618	$n =$	.0655	.8363	.9345	1.0000	1.0982
						$2g =$	.1384	$g =$	.0692	1.1673	1.2655	1.3310	1.4292
						$G =$	.1020	$G =$	.1020				
									1.5708				
.6363	1.0982	.5708	.8363	3.1416									
.2398	1.1673	.3053	1.4292	3.1416									
3.1416 3.1416 3.1416 3.1416 3.1416													

No. 39.

.7854) represents the area of a circle whose diameter is 1: the sum of the four differences next greater ($4n = .2618$) is equal to one-third of that amount; the term in the upper left-hand corner of the square (1.0000) represents the diameter of the circle; and, as should be expected, the sum of every line in any direction and the sum of the four corners of every quadrilateral, whether rectangular or rhomboid, equals 3.1416, the circumference of the circle.

The second example is more complex because it represents more. The sum of the two differences in the middle of each

SQUARE OF THE CIRCLE.				3.1416				DIFFERENCES.		SERIES.			
1.0000	.0193	1.5740	.5483	3.1416		$2a =$	-.0064	$a =$	-.0032	-.0032	.0193	.1331	.1556
						$8d =$	.1800	$d =$	.0025	.5483	.5708	.6846	.7071
1.4377	.6846	.8637	.1555	3.1416		$4n =$	.4552	$n =$	.1138	.8637	.8862	1.0000	1.0225
						$2g =$	.7854	$g =$	.3927	1.4152	1.4377	1.5515	1.5740
						$G =$	.1566	$G =$	.1566				
									1.5708				
-.0032	1.0225	.5708	1.5515	3.1416									
.7071	1.4152	.1331	.8862	3.1416									
3.1416 3.1416 3.1416 3.1416 3.1416													

No. 40.

half of the series ($2g$) equals .7854, the area of a circle whose diameter is 1; no other difference is significant, except that the first term ($a = -.0032$) is less than zero, and being a minus quantity must be subtracted instead of added whenever included in any sum. But in the square no less than three of the terms are significant: The number in the upper left-hand corner (1.0000) represents the diameter of a circle, as before; the opposite number (the one in the lower right-hand corner, .8862) represents an equivalent square, that is to say the side of a square equivalent in area to a circle whose diameter is 1; the lower left-hand corner (.7071) represents the side of the greatest square that can be inscribed in that same circle; and finally, the sum of every line and quadrilateral equals 3.1416, the circumference of the circle.

SMYTH'S THEOREM.

This paper will be closed with a theorem which, while it may not be new, is not taught in the schools as one of the interesting and instructive features of mathematics. It is a principle that upon careful inspection must be acknowledged as a truth; yet it is not sufficiently self-evident to be called an axiom. The proposition is this:

THEOREM.—*The sum of any number of terms (quantities) is equal to the sum of the products of the several terms diminished each by the preceding term and multiplied by the number of terms following that difference.*

The principle is not only true of any line of any perfect square but of any number of numbers whatever, taken in any order, and the numbers may be above zero or below, or mixed in any manner. Before presenting a working formula a few illustrations will be presented by way of demonstration.

The sum of any set of four numbers is equal to four times the first number, plus three times the second minus the first, plus twice the third minus the second, plus the fourth minus the third. When the number to be subtracted is greater than the minuend, then the product of the difference between the two numbers is to be subtracted in the addition. If the numbers be taken in numerical order, the smallest first, then the sum of a series of four numbers is equal to four times the first, plus three times the difference between the first and second, plus twice the difference between the second and third, plus the difference between the third and the fourth.

$$\begin{array}{r} 1 \times 4 = 4 \\ 13 \times 3 = 39 \\ 8 \times 2 = 16 \\ 5 \times 1 = 5 \\ \hline 27 \quad 64 \end{array}$$

For example, take the numbers 1, 14, 22, 27, as in the first line of square No. 25. The first and lowest number in the line is 1; four times that is 4. The next number is 14; the difference between that and 1 is 13; three times 13 is 39. The next difference is 8; twice that is 16. Finally, the difference between 22 and 27 is 5; once that is 5. The sum of these differences equals 27, the last number taken; the sum of the products equals 64, equal to the sum of the numbers.

SQUARE.				
10	36	30	24	$10 \times 4 = 40$
31	23	11	35	$26 \times 3 = 78$
20	26	40	14	$-6 \times 2 = -12$
39	15	19	27	$-6 \times 1 = -6$
				24 100

Again, take the numbers 10, 36, 30, 24, as in the above square, in the order in which they occur in the line. The first is 10, which, multiplied by 4 equals 40; the difference between 10 and 36 is 26, which multiplied by 3 equals 78; the next difference is 6 minus, which multiplied by 2 gives 12 to be subtracted; the final difference is 6, also to be subtracted. The sum of the differences is 24, the last number taken; the sum of the products is 100, equal to the sum of the numbers in the line.

Several other examples are here given of the same problem performed in various ways, but always ending in the same result, namely: The sum of the differences is always equal to the last number taken in the operation; the sum of the products is always equal to the total of all the numbers taken. The examples above given can undoubtedly be understood by inspection without further elucidation.

$\begin{array}{r} 10 - 0 = 10 \\ 24 - 10 = 14 \\ 30 - 24 = 6 \\ 36 - 30 = 6 \\ \hline \end{array}$	$\begin{array}{r} 10 \times 4 = 40 \\ 14 \times 3 = 42 \\ 6 \times 2 = 12 \\ 6 \times 1 = 6 \\ \hline 36 \quad 100 \end{array}$	$\begin{array}{r} 36 - 0 = 36 \\ 30 - 36 = -6 \\ 24 - 30 = -6 \\ 10 - 24 = -14 \\ \hline \end{array}$	$\begin{array}{r} 36 \times 4 = 144 \\ -6 \times 3 = -18 \\ -6 \times 2 = -12 \\ -14 \times 1 = -14 \\ \hline 10 \quad 100 \end{array}$
$\begin{array}{r} 24 - 0 = 24 \\ 30 - 24 = 6 \\ 36 - 30 = 6 \\ 10 - 36 = -36 \\ \hline \end{array}$	$\begin{array}{r} 24 \times 4 = 96 \\ 6 \times 3 = 18 \\ 6 \times 2 = 12 \\ -36 \times 1 = -36 \\ \hline 10 \quad 100 \end{array}$	$\begin{array}{r} 36 - 0 = 36 \\ 10 - 36 = -26 \\ 24 - 10 = 14 \\ 30 - 24 = 6 \\ \hline \end{array}$	$\begin{array}{r} 36 \times 4 = 144 \\ -26 \times 3 = -78 \\ 14 \times 2 = 28 \\ 6 \times 1 = 6 \\ \hline 30 \quad 100 \end{array}$

If, after arranging a series of numbers in four sets preparatory to constructing a magic square, we take the successive differences in the first set and multiply them in order by 4, 3, 2, and 1, respectively, and to the products add the differences between the initials of the sets multiplied in consecutive order by 3, 2, and 1, we obtain as sum of the differences the highest

				$10 - 0 = 10$	$1 \times 4 = 40$
				$11 - 10 = 1$	$1 \times 3 = 3$
				$14 - 11 = 3$	$3 \times 2 = 6$
				$15 - 14 = 1$	$1 \times 1 = 1$
10	11	14	15	$19 - 10 = 9$	$9 \times 3 = 27$
19	20	23	24	$26 - 19 = 7$	$7 \times 2 = 14$
26	27	30	31	$35 - 26 = 9$	$9 \times 1 = 9$
35	36	39	40		
					40 100

number of the entire series, and as sum of the products the sum of any line or subsquare in a square made from that series. This is true of any series and of a square of any size.

One more illustration ought to suffice: Let it be required to add according to the theorem some numbers of which we know the sum, say 15, 11, —8.5, 19, 26.5, 37, no matter what the numbers are or the order in which they occur. The first

15	— 0	=	15	15	$\times 6 =$	90
11	— 15	=	— 4	— 4	$\times 5 =$	— 20
— 8.5	— 11	=	— 19.5	— 19.5	$\times 4 =$	— 78
19	— — 8.5	=	27.5	27.5	$\times 3 =$	82.5
26.5	— 19	=	7.5	7.5	$\times 2 =$	15
37	— 26.5	=	10.5	10.5	$\times 1 =$	10.5
				37		100

difference (which is the first number, the difference being the difference between itself and 0) is to be multiplied by 6, as there are six numbers to be added; the next difference, minus 4 (11—15), is to be multiplied by 5 and subtracted; the next difference is minus 19.5, which is to be multiplied by 4 and subtracted; the other differences, 27.5, 7.5, and 10.5, are to be multiplied in their order by 3, 2, and 1, and added. The entire sum of the products equals 100, as the numbers do; and the differences sum up 37, which is equal to the last number taken.

The foregoing principle may be stated as follows: The sum of any number of terms is equal to the lowest or smallest term multiplied by the number of terms, plus the difference between the lowest term and the second in order multiplied by the number of terms less 1, plus the difference between the second and third multiplied by the number of terms less 2, plus the next difference multiplied by the number of terms less 3, and so on to the end or highest term. The sum of the differences will be equal to the highest term and the sum of the products will be equal to the sum of all the terms.

Again, when a series of numbers is arranged in sets of equal length, the sum of the several differences in the first set, each multiplied in order by the number of terms between that difference and the end, plus the differences between the initials of the several sets multiplied by the number of the difference in order beginning with the last, equals the sum of an average

set of that series; and the sum of the several differences taken equals the highest term of that series.

We may put these principles into intelligible working formulæ by adopting symbols for the several quantities expressed. For the first, let a, b, c , etc., to p, q , represent the several terms of a series, p and q being the last two; let n be the number of terms in the series and let S be the total sum. Then a general formula will be:

$$S = an + (b-a)(n-1) + (c-b)(n-2) + (d-c)(n-3) \dots (q-p)(n-(n-1)).$$

Expanding,

$$S = an + bn - an - b + a + cn - bn - 2c + 2b + dn - cn - 3d + 3c \dots + q.$$

Canceling and collecting,

$$S = a + b + c + d \dots + q.$$

q. e. d.

Again, in a series for a magic square, when it becomes necessary to write the series in n sets of n terms each, let Aa, Ab, Ac , to Aq , represent the first set; let Ba, Bb, Bc , etc., represent the second set; Ca, Cb, Cc , etc., represent the third set, and so on; let n be the number of terms in a line and S equal the sum of the numbers in a line; then the formula will be:

$$S = Sn = Aan + (Ab - Aa)(n-1) + (Ac - Ab)(n-2) \dots + (Aq - Ap)(n - (n-1)) + \\ Ban + (Bb - Ba)(n-1) + (Bc - Bb)(n-2) \dots + (Bq - Bp)(n - (n-1)) + \\ Can + (Cb - Ca)(n-1) + (Cc - Cb)(n-2) \dots + (Cq - Cp)(n - (n-1)) + \\ \dots + (Qb - Qa)(n-1) + (Qc - Qb)(n-2) \dots + (Qq - Qp)(n - (n-1))$$

Expanding, the equation will be:

$$S = Aan + Bbn - Aan - Bb + Aa + Ccn - Bbn - 2Cc + 2Bb \dots + Qq.$$

Simplifying,

$$S = Aa + Bb + Cc \dots + Qq,$$

which is a self-evident fact.

An apparently simpler formula, though not so explicit unless well understood, would be to let d°, d', d'' , etc., to d^n , represent the differences between the successive terms of any set (d° being the first term or the difference between itself and zero; and let D', D'' , etc., to D^n , represent the differences between the initials of the several sets; then the formula will be:

$$S = d^{\circ}n \pm (D' + d')(n-1) \pm (D'' + d'')(n-2) \dots \pm (D^n + d^n)(n - (n-1))$$

Nothing, however, is gained by this formula as it is irreducible. It must be known and expressed in the formula whether any term is greater or less than the preceding term,

and, therefore, whether that difference when increased by its proper multiplier is to be added or subtracted. The formula does not specify this. It must be understood that every term is a minuend from which the preceding term is to be subtracted. When the preceding term is the greater, then the difference becomes a minus quantity and is to be subtracted in the general addition instead of added. When no term precedes, as in the case of the first term, then 0 is to be subtracted and the term itself becomes a difference.